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RADEMACHER AVERAGES ON NONCOMMUTATIVE SYMMETRIC SPACES

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Abstract. Let E be a separable (or the dual of a separable) symmetric function space, let M be a semifinite von Neumann algebra and let $E(M)$ be the associated noncommutative function space. Let $(\varepsilon_k)_{k\geq 1}$ be a Rademacher sequence, on some probability space Ω. For finite sequences $(x_k)_{k\geq 1}$ of $E(M)$, we consider the Rademacher averages $\sum_k \varepsilon_k \otimes x_k$ as elements of the noncommutative function space $E(L^\infty(\Omega)\overline{\otimes}M)$ and study estimates for their norms $\|\sum_k \varepsilon_k \otimes x_k\|_E$ calculated in that space. We establish general Khintchine type inequalities in this context. Then we show that if E is 2-concave, $\|\sum_k \varepsilon_k\otimes x_k\|_E$ is equivalent to the infimum of $\|(\sum y_k^* y_k)^{\frac{1}{2}}\| + \|(\sum z_k z_k^*)^{\frac{1}{2}}\|$ over all y_k, z_k in $E(M)$ such that $x_k = y_k + z_k$ for any $k\geq 1$. Dual estimates are given when E is 2-convex and has a non trivial upper Boyd index. In this case, $\|\sum_k \varepsilon_k \otimes x_k\|_E$ is equivalent to $\|(\sum x_k^* x_k)^{\frac{1}{2}}\| + \|(\sum x_k x_k^*)^{\frac{1}{2}}\|$. We also study Rademacher averages $\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}$ for doubly indexed families $(x_{ij})_{i,j}$ of $E(M)$.

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1. Introduction

The purpose of this paper is to study Rademacher averages on noncommutative symmetric spaces, in connection with some recent developments of noncommutative Khintchine inequalities. Throughout we let E be a symmetric Banach function space on $(0,\infty)$ (see [15]). Let (M,τ) be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. For any x belonging to the space $\widetilde{M}$ of all τ-measurable operators, let $\mu(x)\colon t>0\mapsto \mu_t(x)$ denote the singular value function of x (see [11]). Then the noncommutative symmetric space $E(M)$ is the space of all $x\in\widetilde{M}$ such that $\mu(x)\in E$, equipped with the norm

$$\|x\|_{E(M)} \,=\, \|\mu(x)\|_E.$$

We refer to [15, 16] for general facts on symmetric function spaces and to [24, 31, 6, 7, 8, 14] for a thorough study of $E(M)$-spaces and their properties. We note that in the case when $E = L^p(0,\infty)$, the space $E(M) = L^p(M)$ is the usual noncommutative L^p-space associated with M (see e.g. [11], [29] and the references therein).

In general we will simply let $\|\ \|_E$ (instead of $\|\ \|_{E(M)}$) denote the norm on $E(M)$ if there is no risk of confusion.

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Let $(\Sigma, d\mu)$ be a localizable measure space and consider the commutative von Neumann algebra $L^\infty(\Sigma)$. In the sequel, we will always consider the von Neumann algebra tensor product $L^\infty(\Sigma)\overline{\otimes}M$ as equipped with the trace $d\mu\otimes\tau$. This gives rise to noncommutative spaces $E(L^\infty(\Sigma)\overline{\otimes}M)$. Note that $L^p(L^\infty(\Sigma)\overline{\otimes}M)$ coincides with the Bochner space $L^p(\Sigma; L^p(M))$ for any $p\geq 1$. However $E(L^\infty(\Sigma)\overline{\otimes}M)$ is not a Bochner $E(M)$-valued space in general.

Consider the compact group $\Omega = \{-1,1\}^\infty$, equipped with its normalized Haar measure dm. For $k\geq 1$, we let $\varepsilon_k\colon \Omega\to\{-1,1\}$ be the Rademacher functions defined by letting $\varepsilon_k(\Theta) = \theta_k$ for any $\Theta = (\theta_k)_{k\geq 1}\in\Omega$. Let $x_1,\ldots,x_n$ be a finite family of $E(M)$. We will consider two Rademacher averages of the x_k's. First we let

$$\Bigl\Vert\sum_k \varepsilon_k\otimes x_k\Bigr\Vert_{\mathrm{Rad}(E)} \,=\, \Bigl(\int_\Omega \Bigl\Vert\sum_k \varepsilon_k(\Theta)x_k\Bigr\Vert_{E(M)}\, dm(\Theta)\Bigr)$$

be the norm of the sum $\sum_k \varepsilon_k\otimes x_k$ in the Bochner space $L^1(\Omega; E(M))$. Next we note that each $\varepsilon_k\otimes x_k$ belongs to the noncommutative space $E(L^\infty(\Omega)\overline{\otimes}M)$ and we let

$$\Bigl\Vert\sum_k \varepsilon_k\otimes x_k\Bigr\Vert_E$$

be the norm of their sum $\sum_k \varepsilon_k\otimes x_k$ in the latter space.

Classical commutative or noncommutative Khintchine inequalities involve the 'classical' Rademacher averages expressed by $\Vert\ \Vert_{\mathrm{Rad}(E)}$ (see e.g. [16, 17, 18, 29, 19]). In this paper we will be interested in Khintchine inequalities regarding the averages expressed by $\Vert\ \Vert_E$. In general, the averages $\Vert\ \Vert_{\mathrm{Rad}(E)}$ and $\Vert\ \Vert_E$ are not equivalent, see Section 4 for more on this topic.

We let p_E and q_E denote the Boyd indices of E (see [16, Def. 2.b.1]).

In the case when $M = L^\infty(\Sigma)$ is a commutative von Neumann algebra, it follows from [16, Prop. 2.d.1] and its proof that whenever $q_E<\infty$, we have an equivalence

$$\Bigl\Vert\sum_k \varepsilon_k\otimes x_k\Bigr\Vert_E \,\approx\, \Bigl\Vert\Bigl(\sum_k |x_k|^2\Bigr)^{\frac12}\Bigr\Vert_{E(\Sigma)}$$

for finite families $(x_k)_k$ of $E(\Sigma)$. The main purpose of this paper is to establish noncommutative versions of this theorem.

To express them, we first note that whenever $A(\cdot\,)$ and $B(\cdot\,)$ are two quantities depending on a parameter ω, we will write $A(\omega)\lesssim B(\omega)$ provided that there is a positive constant K such that $A(\omega)\leq KB(\omega)$ for any ω. Then we write $A(\omega)\approx B(\omega)$ when we both have $A(\omega)\lesssim B(\omega)$ and $A(\omega)\gtrsim B(\omega)$.

Given an arbitrary M and a finite family $(x_k)_k$ of $E(M)$, we set

$$\Vert(x_k)_k\Vert_c \,=\, \Bigl\Vert\Bigl(\sum_k x_k^*x_k\Bigr)^{\frac12}\Bigr\Vert_{E(M)} \quad\text{and}\quad \Vert(x_k)_k\Vert_r \,=\, \Bigl\Vert\Bigl(\sum_k x_kx_k^*\Bigr)^{\frac12}\Bigr\Vert_{E(M)}. \tag{1.1}$$

Next we let

$$\Vert(x_k)_k\Vert_{\max} \,=\, \max\Bigl\{\Vert(x_k)_k\Vert_c\,,\,\Vert(x_k)_k\Vert_r\Bigr\}$$

and

$$\Vert(x_k)_k\Vert_{\inf} \,=\, \inf\Bigl\{\Vert(y_k)_k\Vert_c + \Vert(z_k)_k\Vert_r\Bigr\},$$

where the infimum runs over all finite families $(y_k)_k$ and $(z_k)_k$ in $E(M)$ such that $x_k = y_k + z_k$ for any $k \geq 1$.

The classical noncommutative Khintchine inequalities [17, 18, 29] say that if $2 \leq p < \infty$, then we have an equivalence

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(L^p)} \approx \big\| (x_k)_k \big\|_{\max} \tag{1.2}$$

for finite families $(x_k)_k$ of $L^p(M)$, whereas if $1 \leq p \leq 2$, we have an equivalence

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(L^p)} \approx \big\| (x_k)_k \big\|_{\inf}. \tag{1.3}$$

Furthermore it is shown in [19] that (1.2) remains true when L^p is replaced by any E which is 2-convex and q-concave for some $q < \infty$, and that (1.3) remains true when L^p is replaced by any E which is 2-concave.

The main result of this paper is the following.

Theorem 1.1. *Assume that E is separable, or that E is the dual of a separable symmetric function space.*

(1) *If $q_E < \infty$, then we have*

$$\big\| (x_k)_k \big\|_{\inf} \lesssim \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_E \tag{1.4}$$

for finite families $(x_k)_k$ of $E(M)$.

(2) *If $q_E < \infty$ and $p_E > 1$, then we have*

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_E \lesssim \big\| (x_k)_k \big\|_{\max} \tag{1.5}$$

for finite families $(x_k)_k$ of $E(M)$.

This theorem will be proved in Section 3. The preceding Section 2 contains preparatory and preliminary results. Section 4 is devoted to examples and illustrations. In Corollary 4.2, we show that the Rademacher averages $\big\|\sum_k \varepsilon_k \otimes x_k\big\|_E$ are equivalent to $\|(x_k)_k\|_{\inf}$ if E is 2-concave, and that they are equivalent to $\|(x_k)_k\|_{\max}$ if E is 2-convex and $q_E < \infty$. These are analogs of the main results of [19]. Also we discuss equivalence between $\big\|\sum_k \varepsilon_k \otimes x_k\big\|_E$ and the classical averages $\big\|\sum_k \varepsilon_k \otimes x_k\big\|_{\mathrm{Rad}(E)}$ and show that Theorem 1.1 is in some sense optimal. In Section 5, we study double sums

$$\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij},$$

regarded as elements of $E(L^\infty(\Omega)\overline{\otimes}L^\infty(\Omega)\overline{\otimes}M)$. We extend Theorem 1.1 and Corollary 4.2 to this setting.

We mention (as an open problem) that we do not know if the second part of Theorem 1.1 remains true without assuming that $p_E > 1$.

2. Preliminaries and background

In this section, we assume that E is a fully symmetric function space on $(0,\infty)$ (in the sense of [7]). We do not make any assumption on its Boyd indices. We let (M,τ) and (N,σ) denote arbitrary semifinite von Neumann algebras.

Given $1\leq p\leq q\leq\infty$, we will write

$$E\in \mathrm{Int}(L^p,L^q)$$

provided that the following interpolation property holds: Whenever T is a linear operator from $L^p(0,\infty)+L^q(0,\infty)$ into itself which is bounded from $L^p(0,\infty)$ into $L^p(0,\infty)$ and from $L^q(0,\infty)$ into $L^q(0,\infty)$, then T maps E into itself. We recall that in this case, the resulting operator

$$T\colon E\longrightarrow E$$

is automatically bounded. We refer e.g. to [15, I., Sect. 4] or [13] for the necessary background on interpolation. We recall that if $E\in\mathrm{Int}(L^p,L^q)$, then we have

$$p\leq p_E\leq q_E\leq q. \tag{2.1}$$

We also note that the fully symmetric assumption on E is equivalent to the property $E\in \mathrm{Int}(L^1,L^\infty)$ (see e.g. [15, II, Thm. 4.3]). Further, any symmetric function space which is either separable, or is the dual of a separable symmetric function space, is automatically fully symmetric.

Throughout the paper, we will use the following fundamental result from [7].

Proposition 2.1. *Assume that $E\in\mathrm{Int}(L^p,L^q)$ and let*

$$T\colon L^p(M)+L^q(M)\longrightarrow L^p(N)+L^q(N)$$

be any linear operator such that

$$T\colon L^p(M)\longrightarrow L^p(N)\qquad and\qquad T\colon L^q(M)\longrightarrow L^q(N)$$

are bounded. Then T maps $E(M)$ into $E(N)$ and the resulting operator $T\colon E(M)\to E(N)$ is bounded. Moreover we have an estimate

$$\|T\colon E(M)\to E(N)\|\leq C\,\max\bigl\{\|T\colon L^p(M)\to L^p(N)\|\,,\,\|T\colon L^q(M)\to L^q(N)\|\bigr\}$$

for some constant C not depending on either M, N, or T.

The norms introduced in (1.1) are related to matrix representations, as follows. Let $n\geq 1$ be an integer and consider the von Neumann algebra $M_n(M)$ equipped with the trace $tr\otimes\tau$ (here tr is the usual trace on M_n). In the sequel we will write E_{ij} for the usual matrix units of M_n. For any $x\in E(M)$, we have

$$|E_{ij}\otimes x| = E_{jj}\otimes|x|,$$

hence $E_{ij}\otimes x\in E(M_n(M))$, with $\|E_{ij}\otimes x\|_E=\|x\|_E$. We deduce that the space $E(M_n(M))$ can be algebraically identified with the space $M_n\otimes E(M)$ of $n\times n$ matrices with entries in $E(M)$. Then for any $x_1,\ldots,x_n$ in $E(M)$, we have

$$\Bigl|\sum_{k=1}^n E_{k1}\otimes x_k\Bigr| = E_{11}\otimes\Bigl(\sum_{k=1}^n x_k^*x_k\Bigr)^{\frac{1}{2}},$$

hence

$$(2.2)\qquad \Big\|\Big(\sum_{k=1}^n x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_{E(M)} \,=\, \Big\|\sum_{k=1}^n E_{k1}\otimes x_k\Big\|_{E(M_n(M))} \,=\, \left\|\begin{bmatrix} x_1 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ \vdots & \vdots & & \vdots\\ x_n & 0 & \cdots & 0\end{bmatrix}\right\|_{E(M_n(M))}.$$

Likewise, we have

$$\Big\|\Big(\sum_{k=1}^n x_kx_k^*\Big)^{\frac{1}{2}}\Big\|_{E(M)} \,=\, \Big\|\sum_{k=1}^n E_{1k}\otimes x_k\Big\|_{E(M_n(M))} \,=\, \left\|\begin{bmatrix} x_1 & \cdots & x_n\\ 0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{bmatrix}\right\|_{E(M_n(M))}.$$

Let $\mathrm{Col}_n\colon M_n(M)\to M_n(M)$ be the natural projection onto the 'column subspace' of $M_n(M)$, taking $E_{i1}\otimes x$ to itself for any $i\geq 1$ and taking $E_{ij}\otimes x$ to 0 whenever $j\geq 2$. Alternatively, Col_n is the right multiplication $z\mapsto zc$, where $c=E_{11}\otimes 1$. Since $\mu_t(zc)\leq \mu_t(z)\|c\|_M=\mu_t(z)$ for any $z\in\widetilde{M}$ and any $t>0$, the mapping $\mathrm{Col_n}$ extends to a contractive projection on $E(M_n(M))$, that is,

$$(2.3)\qquad \big\|\mathrm{Col}_n\colon E(M_n(M))\longrightarrow E(M_n(M))\big\|\leq 1.$$

We record for further use the following straightforward consequence of the latter observation and Proposition 2.1.

Lemma 2.2. *Assume that $E\in\mathrm{Int}(L^p,L^q)$. There is a constant C verifying the following property. Let $n\geq 1$ be an integer and let $T\colon L^p(M)^n+L^q(M)^n\to L^p(N)+L^q(N)$ be a linear map such that for r equal to either p or q, we have*

$$\big\|T(x_1,\ldots,x_n)\big\|_r \,\leq\, \Big\|\Big(\sum_{k=1}^n x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_r,\qquad x_1,\ldots,x_n\in L^r(M).$$

Then we have

$$\big\|T(x_1,\ldots,x_n)\big\|_E \,\leq\, C\Big\|\Big(\sum_{k=1}^n x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_E,\qquad x_1,\ldots,x_n\in E(M).$$

Let $n\geq 1$ be an integer and let $\big(E(M)^n,\|\ \|_{\inf}\big)$ (resp. $\big(E(M)^n,\|\ \|_{\max}\big)$) denote the product space $E(M)^n$ of all n-tuples $(x_1,\ldots,x_n)$ of $E(M)$ equipped with the norm $\|(x_k)_k\|_{\inf}$ (resp. $\|(x_k)_k\|_{\max}$). Let E' denote the Köthe dual of E and recall that when E is separable, then $E'=E^*$ (see e.g. [15, p. 102]).

Proposition 2.3. *Let $n\geq 1$ be an integer.*

(1) *If E is separable, then we both have*

$$\big(E(M)^n,\|\ \|_{\inf}\big)^* \,=\, \big(E'(M)^n,\|\ \|_{\max}\big)\quad \textit{and}\quad \big(E(M)^n,\|\ \|_{\max}\big)^* \,=\, \big(E'(M)^n,\|\ \|_{\inf}\big)$$

isometrically.

(2) *If E is separable, or if E is the dual of a separable symmetric space, then for any $x_1,\ldots,x_n$ in $E(M)$, we have*

$$\big\|(x_k)_k\big\|_{\rm inf} \,=\, \sup\big\{\big\|(sx_ks)_k\big\|_{\rm inf}\,:\, s \textit{ is a selfadjoint projection from } M,\ \tau(s)<\infty\,\big\}.$$

Proof. Assume that E is separable. Then it follows from [8, p. 745] that $E(M_n(M))^* = E'(M_n(M))$. Using (2.3) and its row counterpart, this implies that

$$\big(E(M)^n, \|\ \|_c\big)^* \,=\, \big(E'(M)^n, \|\ \|_r\big) \quad\text{and}\quad \big(E(M)^n, \|\ \|_r\big)^* \,=\, \big(E'(M)^n, \|\ \|_c\big)$$

isometrically. Then part (1) of the proposition follows at once, using standard duality principles (see e.g. [15, Thm. I.3.1]).

Now take $x_1,\ldots,x_n$ in $E(M)$. By the preceding point, there exist $y_1,\ldots,y_n$ in $E'(M)$ such that $\|(y_k)_k\|_{\rm max}=1$ and

$$\big\|(x_k)_k\big\|_{\rm inf} \,=\, \sum_{k=1}^n \tau(x_ky_k).$$

Let $\varepsilon>0$. For every $k=1,\ldots,n$, there exists a selfadjoint projection s_k from M such that $\tau(s_k)<\infty$ and

$$\big|\tau(sx_ksy_k)-\tau(x_ky_k)\big| \,\leq\, \frac{\varepsilon}{n}$$

for every $s\geq s_k$. Indeed, this follows from [5, Prop. 2.5] (the latter proposition holds true for general semifinite von Neumann algebras). Set $s=\bigvee\limits_{1\leq k\leq n} s_k$. Clearly $\tau(s)<\infty$ and we have

$$\Big|\sum_{k=1}^n \tau(sx_ksy_k)\Big| \,\geq\, \big\|(x_k)_k\big\|_{\rm inf} - \varepsilon.$$

Furthermore,

$$\Big|\sum_{k=1}^n \tau(sx_ksy_k)\Big| \,\leq\, \big\|(sx_ks)_k\big\|_{\rm inf}\big\|(y_k)_k\big\|_{\rm max} = \big\|(sx_ks)_k\big\|_{\rm inf}$$

by part (1) of this proposition. Hence $\|(sx_ks)_k\|_{\rm inf}\geq \|(x_k)_k\|_{\rm inf}-\varepsilon$. This shows the non trivial inequality of part (2) in the case when E is separable. The proof in the case when E is the dual of a separable symmetric space is similar, by applying part (1) to the predual of E. ☐

Remark 2.4. Suppose that E is separable. Applying [8, p. 745] as above, we have

$$E(L^\infty(\Omega)\overline{\otimes}M)^* = E'(L^\infty(\Omega)\overline{\otimes}M). \tag{2.4}$$

Let $P\colon L^2(\Omega)\to L^2(\Omega)$ be the orthogonal projection onto the closed linear span of the ε_k's. For any $1<r<\infty$, $L^r(M)$ is a K-convex Banach space in the sense of [26] (see also [22]). Indeed, $L^r(M)$ has a non trivial type (see e.g. [29, Cor. 5.5]). Thus the linear map $P\otimes I_{L^r(M)}$ extends to a bounded projection $L^r(\Omega;L^r(M))\to L^r(\Omega;L^r(M))$.

Assume now that $p_E>1$ and $q_E<\infty$. Owing to the relations $p_{E'}=\frac{q_E}{q_E-1}$ and $q_{E'}=\frac{p_E}{p_E-1}$ [16, Prop. 2.b.2], we have $p_{E'}>1$ and $q_{E'}<\infty$ as well. Then Boyd's Theorem (see e.g. [13, Thm. 7.3]) ensures that $E\in{\rm Int}(L^p,L^q)$ and $E'\in{\rm Int}(L^p,L^q)$ for some $1<p\leq q<\infty$.

Applying the boundedness of $P \otimes I_{L^r(M)}$ above with $r = p$ and $r = q$, we deduce by interpolation that $P \otimes I_E$ and $P \otimes I_{E'}$ extend to bounded projections

$$E(L^\infty(\Omega)\overline{\otimes}M) \longrightarrow E(L^\infty(\Omega)\overline{\otimes}M) \qquad \text{and} \qquad E'(L^\infty(\Omega)\overline{\otimes}M) \longrightarrow E'(L^\infty(\Omega)\overline{\otimes}M),$$

respectively. Combining with the duality identification (2.4), this yields equivalence properties

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \approx \sup\Big\{\Big|\sum_k \tau(x_k y_k)\Big| \,:\, y_k \in E',\ \Big\|\sum_k \varepsilon_k \otimes y_k\Big\|_{E'} \leq 1\Big\}$$

and

$$\Big\|\sum_k \varepsilon_k \otimes y_k\Big\|_{E'} \approx \sup\Big\{\Big|\sum_k \tau(x_k y_k)\Big| \,:\, x_k \in E,\ \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{E} \leq 1\Big\}.$$

We refer to [9, Thm. 1.3] for a slightly more precise result when E is reflexive.

Let $E^{(2)}$ denote the 2-convexification of E. We will use the following well-known Cauchy-Schwarz inequality. For any $g, h \in E^{(2)}(M)$, we have $gh \in E(M)$ and

$$\|gh\|_{E(M)} \leq \|g\|_{E^{(2)}(M)}\|h\|_{E^{(2)}(M)}. \tag{2.5}$$

Indeed, this inequality follows from [11, Thm 4.2].

We will use Hardy spaces associated to symmetric function spaces. We start with a general definition. Assume here that (N, σ) is a finite von Neumann algebra and let $H^\infty(N) \subset N$ be a finite subdiagonal algebra in the sense of [29, Section 8]. Recall that for any $1 \leq p < \infty$, the associated Hardy space $H^p(N)$ is defined as the closure of $H^\infty(N)$ into $L^p(N)$ and that we actually have $H^p(N) = H^1(N) \cap L^p(N)$ (see [30] and [20, (3.1)]). Note that since N is finite, we have a continuous inclusion $E(N) \subset L^1(N)$. Then we let

$$H^E(N) = H^1(N) \cap E(N),$$

that we regard as a subspace of $E(N)$ equipped with the induced norm. It is plain that $H^E(N) \subset E(N)$ is closed.

We clearly have $E^{(2)}(N) \subset L^2(N)$, and hence $H^{E^{(2)}}(N) \subset H^2(N)$. Owing to the fact that the product of two elements of $H^2(N)$ belongs to $H^1(N)$, we deduce that for any $x, y \in H^{E^{(2)}}(N)$, the product xy belongs to $H^E(N)$. As a converse to this embedding and its companion estimate (2.5), we have the following factorization property.

Proposition 2.5. *For every f in $H^E(N)$ and for every $\varepsilon > 0$, there exist $g, h \in H^{E^{(2)}}(N)$ such that $f = gh$ and*

$$\|g\|_{E^{(2)}}\|h\|_{E^{(2)}} \leq (1+\varepsilon)\|f\|_E.$$

Proof. In the case when $E = L^p$, this result is due to Marsalli-West [20, Thm. 4.3]. In turn, the latter result relies on [20, Thm. 4.2]. It is not hard to adapt the proof of these two theorems to the above general case. An alternative route consists in adapting the proof of [3, Thm. 3.4]. Details are left to the reader. $\square$

Remark 2.6. When dealing with noncommutative Hardy spaces as above, it is usually assumed that the trace σ on N is normalized, i.e. $\sigma(1) = 1$. However the proofs of [20, Thms 4.2 and 4.3] work as well under the mere assumption that $\sigma(1) < \infty$. Likewise, the above Proposition 2.5 holds true whenever $\sigma(1) < \infty$.

In Section 3 we will apply the previous proposition in the following classical context. We let $\mathbb{T}$ denote the unit circle, equipped with Haar measure, and we identify $L^\infty(\mathbb{T})$ with the space of essentially bounded 2π-periodic functions from $\mathbb{R}$ into $\mathbb{C}$ in the usual way. Next let (M,τ) be a finite von Neumann algebra and let

$$N = L^\infty(\mathbb{T})\overline{\otimes}M.$$

Since $L^1(N) = L^1(\mathbb{T}; L^1(M))$ one can define Fourier coefficients on $L^1(N)$ by letting

$$\widehat{f}(j) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-ijt}\, dt \ \in L^1(M), \qquad f\in L^1(N),\ j\in\mathbb{Z}.$$

Then the space

$$H^\infty(N) \,=\, \bigl\{f\in N\,:\,\forall\, j<0,\ \widehat{f}(j)=0\bigr\}$$

is a finite subdiagonal algebra. Note that the resulting Hardy spaces coincide with the vector valued H^p-spaces $H^p(\mathbb{T}; L^p(M))$, for any $1\leq p<\infty$. It is clear that

$$H^E(L^\infty(\mathbb{T})\overline{\otimes}M) = \bigl\{f\in E(L^\infty(\mathbb{T})\overline{\otimes}M)\,:\,\forall\, j<0,\ \widehat{f}(j)=0\bigr\}$$

We conclude this section with a few remarks on Fourier coefficients on $E(L^\infty(\mathbb{T})\overline{\otimes}M)$. For any $j\in\mathbb{Z}$, let F_j be the linear map taking any $f\in L^1(\mathbb{T}; L^1(M))$ to $\widehat{f}(j)$. Then F_j is both a contraction from $L^1(\mathbb{T}; L^1(M))$ into $L^1(M)$ and from $L^\infty(\mathbb{T})\overline{\otimes}M$ into M. Hence by the interpolation Proposition 2.1, it also extends to a bounded operator

$$F_{j,E}\colon E(L^\infty(\mathbb{T})\overline{\otimes}M)\longrightarrow E(M).$$

Indeed there exists a constant C_E (only depending on E) such that

$$\|\widehat{f}(j)\|_{E(M)}\,\leq C_E\,\|f\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)},\qquad f\in E(L^\infty(\mathbb{T})\overline{\otimes}M),\ j\in\mathbb{Z}. \tag{2.6}$$

Lemma 2.7. *For any $f_1,\ldots,f_n$ in $E(L^\infty(\mathbb{T})\overline{\otimes}M)$, and for any $j\in\mathbb{Z}$, we have*

$$\Bigl\|\Bigl(\sum_{k=1}^n \widehat{f_k}(j)^*\widehat{f_k}(j)\Bigr)^{\frac12}\Bigr\|_{E(M)}\,\leq\, C_E\,\Bigl\|\Bigl(\sum_{k=1}^n f_k^*f_k\Bigr)^{\frac12}\Bigr\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)}.$$

Proof. We apply the above results on $M_n(M)$, with

$$f\,=\,\sum_{k=1}^n E_{k1}\otimes f_k.$$

For any j, the Fourier coefficient $\widehat{f}(j)$ is equal to $\sum_k E_{k1}\otimes\widehat{f_k}(j)$. Hence the result follows from (2.6) and (2.2). $\square$

3. Proof of the main result

Throughout this section we let (M,τ) be a semifinite von Neumann algebra. Our aim is to prove Theorem 1.1. We start with an equivalence property which will enable us to deduce the estimation of the Rademacher averages on $E(M)$ from an estimation of a certain lacunary Fourier series.

We set $e_j(t) = e^{ijt}$ for any integer $j \in \mathbb{Z}$ and any $t \in \mathbb{R}$. The left hand-side in the equivalence below is the norm of $\sum_k e_{3^k} \otimes x_k$ in $E(L^\infty(\mathbb{T})\overline{\otimes} M)$. Note that no assumption on either p_E or q_E is made in the next statement.

Proposition 3.1. *Assume that E is fully symmetric. Then we have an equivalence*

$$\Big\|\sum_k e_{3^k} \otimes x_k\Big\|_E \approx \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E$$

for finite families $(x_k)_k$ of $E(M)$.

Proof. For any $k \geq 1$, let $\eta_k(t) = \cos(3^k t)$ and $\widetilde{\eta}_k(t) = \sin(3^k t)$ for $t \in \mathbb{R}$. It is plain that

$$\Big\|\sum_k \eta_k \otimes x_k\Big\|_E \leq \Big\|\sum_k e_{3^k} \otimes x_k\Big\|_E \leq \Big\|\sum_k \eta_k \otimes x_k\Big\|_E + \Big\|\sum_k \widetilde{\eta}_k \otimes x_k\Big\|_E. \tag{3.1}$$

Here the norms are computed in $E(L^\infty(\mathbb{T})\overline{\otimes} M)$. Indeed, the mapping $g(t) \mapsto g(-t)$ is an isometry on the latter space, which proves the first inequality. The second one is obvious.

Now fix an integer $n \geq 1$ and consider the Riesz product

$$K(\Theta, t) = \prod_{k=1}^{n}\big(1 + \varepsilon_k(\Theta)\eta_k(t)\big), \qquad \Theta \in \Omega,\ t \in \mathbb{R}.$$

Note that K is a nonnegative function. For any $A \subset \{1, \ldots, n\}$, set $\varepsilon_A = \prod_{k\in A} \varepsilon_k$ and $\eta_A = \prod_{k\in A} \eta_k$. By convention, $\varepsilon_\emptyset = 1$ and $\eta_\emptyset = 1$. If $A \neq \emptyset$, then the integrals of ε_A and η_A on Ω and $\mathbb{T}$ respectively are equal to 0. Since

$$K(\Theta, t) = \sum_{A\subset\{1,\ldots,n\}} \varepsilon_A(\Theta)\, \eta_A(t), \qquad \Theta \in \Omega, t \in \mathbb{R},$$

this implies that

$$\sup_\Theta \int_0^{2\pi} |K(\Theta, t)|\, \frac{dt}{2\pi} = 1 \qquad \text{and} \qquad \sup_t \int_\Omega |K(\Theta, t)|\, dm(\Theta) = 1.$$

Consequently, one can define two linear contractions

$$T_1\colon L^1(\Omega; L^1(M)) \longrightarrow L^1(\mathbb{T}; L^1(M)) \qquad \text{and} \qquad T_2\colon L^1(\mathbb{T}; L^1(M)) \longrightarrow L^1(\Omega; L^1(M))$$

by letting

$$[T_1(f)](t) = \int_\Omega K(\Theta, t) f(\Theta)\, dm(\Theta), \qquad f \in L^1(\Omega, L^1(M)),$$

and

$$[T_2(g)](\Theta) = \frac{1}{2\pi}\int_0^{2\pi} K(\Theta, t) g(t)\, dt, \qquad g \in L^1(\mathbb{T}, L^1(M)).$$

Moreover $T_2^*\colon L^\infty(\Omega)\overline{\otimes}M \to L^\infty(\mathbb{T})\overline{\otimes}M$ and T_1 coincide on the intersection of their domains. Thus we may define a linear map

$$T\colon L^1(\Omega; L^1(M)) + L^\infty(\Omega)\overline{\otimes}M \longrightarrow L^1(\mathbb{T}; L^1(M)) + L^\infty(\mathbb{T})\overline{\otimes}M$$

extending both of them. Thus by interpolation (using Proposition 2.1), there is a constant $C \geq 1$ (not depending on n) such that

$$\big\| T\colon E(L^\infty(\Omega)\overline{\otimes}M) \longrightarrow E(L^\infty(\mathbb{T})\overline{\otimes}M)\big\| \leq C.$$

For any $k = 1, \ldots, n$ and any $x \in E(M)$, we have

$$T(\varepsilon_k \otimes x) = \sum_{A \subset \{1,\ldots,n\}} \Big(\int_\Omega \varepsilon_A \varepsilon_k \, dm\Big)\, \eta_A \otimes x = \eta_k \otimes x.$$

Hence T maps $\sum_k \varepsilon_k \otimes x_k$ to $\sum_k \eta_k \otimes x_k$ for any $x_1, \ldots, x_n \in E(M)$ and we obtain that

$$\Big\| \sum_k \eta_k \otimes x_k \Big\|_E \,\leq\, C \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_E.$$

A similar result holds with η_k replaced by $\widetilde{\eta_k}$. According to (3.1), this yields the estimate $\lesssim$ in the proposition. The reverse estimate is proved similarly, using the fact (easy to check) that for any $A \subset \{1, \ldots, n\}$ and any $k = 1, \ldots, n$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \eta_A(t)\eta_k(t)\, dt \,=\, \begin{cases} 0 & \text{if } A \neq \{k\}, \\ \frac{1}{2} & \text{if } A = \{k\}. \end{cases}$$

□

Further equivalence properties of the above type are established in [2].

Remark 3.2. The above result can be regarded as an analog of the following classical result of Pisier. For any Banach space and for any $1 \leq r < \infty$, there is an equivalence

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{L^r(\Omega:X)} \,\approx\, \Big\| \sum_k e_{2^k} \otimes x_k \Big\|_{L^r(\mathbb{T};X)}$$

for finite families $(x_k)_k$ of X [25].

The main result of this section is the following noncommutative Paley inequality, which extends [18].

Theorem 3.3. *Assume that E is fully symmetric and that $q_E < \infty$. There is a constant $C \geq 0$ such that for any finite (M, τ), for any $f \in H^E(L^\infty(\mathbb{T})\overline{\otimes}M)$ and for any $n \geq 1$, we have*

$$\big\| \big(\widehat{f}(3^k)\big)_{k=1}^n \big\|_{\inf} \,\leq\, C\, \|f\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)}.$$

Proof. Throughout we let $(m_j)_{j\geq 0}$ be the sequence given by $m_j = 1$ if $j = 3^k$ for some $k \geq 1$, and $m_j = 0$ otherwise. Let $f \in H^E(L^\infty(\mathbb{T})\overline{\otimes}M)$. Applying Proposition 2.5 (and Remark 2.6) with $N = L^\infty(\mathbb{T})\overline{\otimes}M$ and $\varepsilon = 1$, we find $g, h \in H^{E^{(2)}}(L^\infty(\mathbb{T})\overline{\otimes}M)$ such that $f = gh$ and

$$\|g\|_{E^{(2)}} \|h\|_{E^{(2)}} \leq 2\|f\|_E.$$

By analyticity we have

$$\widehat{f}(j) = \sum_{0\leq i\leq j} \widehat{g}(i)\,\widehat{h}(j-i), \qquad j \geq 0.$$

We define

$$y_j = \sum_{\frac{j}{2}<i\leq j} \widehat{g}(i)\,\widehat{h}(j-i) \qquad \text{and} \qquad z_j = \sum_{0\leq i\leq \frac{j}{2}} \widehat{g}(i)\widehat{h}(j-i)$$

for any $j \geq 0$, so that we have decompositions $\widehat{f}(j) = y_j + z_j$ in $E(M)$. Thus it suffices to show that for some constant $C \geq 0$, we have

$$\big\|\big(m_j y_j\big)_{j=0}^{l}\big\|_c \,\leq\, C\,\|f\|_E \qquad \text{and} \qquad \big\|\big(m_j z_j\big)_{j=0}^{l}\big\|_r \,\leq\, C\,\|f\|_E. \tag{3.2}$$

for any $l \geq 1$. Given any integer $j \geq 0$, we define

$$g_j = \sum_{\frac{j}{2}<i\leq j} e_i \otimes \widehat{g}(i), \qquad \text{and} \qquad h_j = \sum_{\frac{j}{2}\leq i\leq j} e_i \otimes \widehat{h}(i)$$

as elements of $H^{E^{(2)}}(L^\infty(\mathbb{T})\overline{\otimes}M)$. Then we have

$$m_j y_j = m_j \sum_{i=0}^{j} \widehat{g_j}(i)\,\widehat{h}(j-i) = \widehat{m_j g_j h}(j)$$

and similarly,

$$m_j z_j = \widehat{m_j g h_j}(j).$$

We shall now concentrate on the first part of (3.2). Applying Lemma 2.7, we deduce from above that

$$\big\|\big(m_j y_j\big)_{j=0}^{l}\big\|_c \,\leq\, C_E \Big\|\Big(\sum_{j=0}^{l}(m_j g_j h)^*(m_j g_j h)\Big)^{\frac{1}{2}}\Big\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)}.$$

Recall that for any u and v in some space of the form $N + L^1(N)$, with $u \geq 0$, we have $(v^*uv)^{\frac{1}{2}} = |u^{\frac{1}{2}}v|$. Consequently,

$$\begin{aligned}\Big\|\Big(\sum_{j=0}^{l}(m_j g_j h)^*(m_j g_j h)\Big)^{\frac{1}{2}}\Big\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)} &= \Big\|\Big(h^*\Big(\sum_{j=0}^{l}(m_j g_j)^*(m_j g_j)\Big)h\Big)^{\frac{1}{2}}\Big\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)}\\ &= \Big\|\Big(\sum_{j=0}^{l}(m_j g_j)^*(m_j g_j)\Big)^{\frac{1}{2}}\cdot h\Big\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)}\\ &\leq \Big\|\Big(\sum_{k=1}^{l} g_{3^k}^* g_{3^k}\Big)^{\frac{1}{2}}\cdot h\Big\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)}\end{aligned}$$

Applying the 'Cauchy-Schwarz inequality' (2.5) on $L^\infty(\mathbb{T})\overline{\otimes}M$, we therefore obtain that

$$\big\|\big(m_j y_j\big)_{j=1}^{l}\big\|_c \,\leq\, C_E\,\|h\|_{E^{(2)}}\Big\|\Big(\sum_{k=0}^{l} g_{3^k}^* g_{3^k}\Big)^{\frac{1}{2}}\Big\|_{E^{(2)}}.$$

For $l \geq 1$ as above, consider the linear map

$$T\colon L^2(\mathbb{T}; L^2(M)) \longrightarrow L^2(M_l \otimes L^\infty(\mathbb{T})\overline{\otimes} M)$$

defined by

$$T(\varphi) = \begin{bmatrix} \varphi_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \varphi_l & 0 & \cdots & 0 \end{bmatrix}, \qquad \text{with} \quad \varphi_k = \sum_{\frac{3^k}{2} < i \leq 3^k} e_i \otimes \widehat{\varphi}(i).$$

It is plain that T is a contraction on L^2. Now let $2 < q < \infty$ and consider $\varphi \in L^q(\mathbb{T}; L^q(M))$. The easy part of the noncommutative Khintchine inequalities on $L^q(M)$ (which follows from the 2-convexity of L^q) yields

$$\|T(\varphi)\|_{L^q(M_l \otimes L^\infty(\mathbb{T})\overline{\otimes} M)} = \Bigl\|\Bigl(\sum_{k=1}^{l} \varphi_k^* \varphi_k\Bigr)^{\frac{1}{2}}\Bigr\|_{L^q(\mathbb{T}; L^q(M))} \leq \Bigl\|\sum_{k=1}^{l} \varepsilon_k \otimes \varphi_k\Bigr\|_{L^q(\Omega \times \mathbb{T}; L^q(M))}.$$

Furthermore there exists a constant K_q (only depending on q) such that

$$\Bigl\|\sum_{k=1}^{l} \theta_k \varphi_k\Bigr\|_{L^q(\mathbb{T}; L^q(M))} \leq K_q \|\varphi\|_{L^q(\mathbb{T}; L^q(M))}$$

for any $\theta_k = \pm 1$ and any $\varphi \in L^q(\mathbb{T}; L^q(M))$. Indeed, $L^q(M)$ is a UMD Banach space, hence the latter estimate follows from the vector-valued Fourier multiplier theory on this class ([23], see also [4]). We deduce that

$$\bigl\|T\colon L^q(\mathbb{T}; L^q(M)) \longrightarrow L^q(M_l \otimes L^\infty(\mathbb{T})\overline{\otimes} M))\bigr\| \leq K_q.$$

We now use interpolation. The space $E^{(2)}$ is 2-convex by nature, and we know that $q_{E^{(2)}} = 2q_E < \infty$. Hence by [13, Thm. 7.3], there exists $2 < q < \infty$ such that $E \in \mathrm{Int}(L^2; L^q)$. Thus by Proposition 2.1, we have

$$\bigl\|T\colon E^{(2)}(L^\infty(\mathbb{T})\overline{\otimes} M) \longrightarrow E^{(2)}(M_l \otimes L^\infty(\mathbb{T})\overline{\otimes} M)\bigr\| \leq K$$

for some constant K not depending on $l \geq 1$. Now observe that $T(g) = \sum_{k=1}^{l} E_{k1} \otimes g_{3^k}$. According to (2.2), this yields

$$\Bigl\|\Bigl(\sum_{k=1}^{l} g_{3^k}^* g_{3^k}\Bigr)^{\frac{1}{2}}\Bigr\|_{E^{(2)}} \leq K \|g\|_{E^{(2)}}.$$

Consequently, we have

$$\bigl\|\bigl(m_j y_j\bigr)_{j=0}^{l}\bigr\|_c \leq KC_E \|g\|_{E^{(2)}} \|h\|_{E^{(2)}} \leq 2KC_E \|f\|_E.$$

This is the first part of (3.2), and the proof of the second one is similar, using the h_j's. $\square$

Proof of Theorem 1.1. We first prove the lower estimate (part (1)). Let $x_1, \ldots, x_n$ in $E(M)$, and let $s \in M$ be a selfadjoint projection with $\tau(s) < \infty$. Consider the finite von Neumann algebra sMs and the analytic polynomial

$$f_s = \sum_{k=1}^{n} e_{3^k} \otimes s x_k s.$$

This is an element of $H^E(L^\infty(\mathbb{T})\overline{\otimes} sMs)$. Hence by Theorem 3.3, we have an estimate

$$\big\|(s x_k s)_k\big\|_{\inf} \,\leq\, C\|f_s\|_E \leq C\Big\|\sum_k e_{3^k} \otimes x_k\Big\|_E.$$

Then applying Proposition 2.3 (2) and Proposition 3.1, we deduce the lower estimate (1.4).

We now turn to the upper estimate (part (2)). We first assume that E is separable. For any $x_1, \ldots, x_n$ in $E(M)$ and $y_1, \ldots, y_n$ in $E'(M)$, we have

$$\Big|\sum_{k=1}^{n} \tau(x_k y_k)\Big| \,\leq\, \big\|(x_k)_{k=1}^n\big\|_{\max}\big\|(y_k)_{k=1}^n\big\|_{\inf}.$$

By assumption $q_E < \infty$ hence $p_{E'} > 1$. Applying the first part of Theorem 1.1 to E', we therefore deduce that

$$\Big|\sum_{k=1}^{n} \tau(x_k y_k)\Big| \,\leq\, C\big\|(x_k)_{k=1}^n\big\|_{\max}\Big\|\sum_{k=1}^{n} \varepsilon_k \otimes y_k\Big\|_{E'}$$

for some constant $C \geq 0$ not depending either on n, the x_k's or the y_k's. Now applying the first equivalence in Remark 2.4 yields the result.

The proof in the case when E is the dual of a separable symmetric space is similar, using the second equivalence in Remark 2.4. □

4. Examples

In the noncommutative setting, it is well-known that the quantities $\|\ \|_{\max}$ and $\|\ \|_{\inf}$ appearing in Theorem 1.1 are not equivalent in general. In Proposition 4.4 below we will show that the latter theorem cannot be improved in general. For the time being we will consider special cases when the estimates (1.4) or (1.5) can be replaced by an equivalence. Throughout we let (M, τ) be an arbitrary semifinite von Neumann algebra, and we assume that E is either separable or is the dual of a separable symmetric space.

Corollary 4.1.

(1) *Assume that* $E \in \mathrm{Int}(L^1, L^2)$. *Then*

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \,\approx\, \big\|(x_k)_k\big\|_{\inf} \tag{4.1}$$

for finite families $(x_k)_k$ *of* $E(M)$.

(2) *Assume* $E \in \mathrm{Int}(L^2, L^q)$ *for some* $q < \infty$. *Then*

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_E \,\approx\, \big\| (x_k)_k \big\|_{\max} \tag{4.2}$$

for finite families $(x_k)_k$ *of* $E(M)$.

Proof. We only prove part (1), the proof of (2) being similar. By the easy part of the noncommutative Khintchine inequalities on $L^1(M)$ (equivalently, by the 2-concavity of L^1), we have

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{L^1(\mathbb{T};L^1(M))} \,\leq\, \Big\| \Big(\sum_k x_k^* x_k \Big)^{\frac{1}{2}} \Big\|_{L^1(M)}.$$

for any finite family $(x_k)_k$ in $L^1(M)$. On the other hand we obviously have

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{L^2(\mathbb{T};L^2(M))} \,=\, \Big(\sum_k \|x_k\|^2_{L^2(M)} \Big)^{\frac{1}{2}} \,=\, \Big\| \Big(\sum_k x_k^* x_k \Big)^{\frac{1}{2}} \Big\|_{L^2(M)}$$

for $x_k \in L^2(M)$. Applying Lemma 2.2, we deduce an estimate

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_E \,\lesssim\, \big\| (x_k)_k \big\|_c$$

for $x_k \in E(M)$. The same holds true with $\| \ \|_r$ instead of $\| \ \|_c$, and these two estimates immediately imply that $\big\| \sum_k \varepsilon_k \otimes x_k \big\|_E \lesssim \big\| (x_k)_k \big\|_{\inf}$.

Our assumption ensures that $q_E \leq 2$, see (2.1). The converse inequality is therefore given by the lower estimate in Theorem 1.1. □

The next results should be compared with [19].

Corollary 4.2.

(1) *If* E *is* 2*-concave or* $q_E < 2$*, then the equivalence property (4.1) holds true.*
(2) *Assume that* $q_E < \infty$. *If* E *is* 2*-convex or* $p_E > 2$*, then the equivalence property (4.2) holds true.*

Proof. This follows from the previous corollary. Indeed by [13, Thm. 7.3], the assumption in (1) ensures that $E \in \mathrm{Int}(L^1, L^2)$ whereas the assumption in (2) ensures that $E \in \mathrm{Int}(L^2, L^q)$ for some $q < \infty$. □

It is well-known that the Rademacher averages $\big\| \sum_k \varepsilon_k \otimes x_k \big\|_E$ studied in the present paper and the 'classical' Rademacher averages $\big\| \sum_k \varepsilon_k \otimes x_k \big\|_{\mathrm{Rad}(E)}$ are not equivalent in general. In fact, rather little is known on the cases when

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_E \,\approx\, \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(E)}, \qquad x_k \in E(M). \tag{4.3}$$

In the commutative setting, we have the following two characterizations at our disposal. First it follows from [16, Prop. 2.d.1] and [1] that

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_E \,\approx\, \Big\| \Big(\sum_k |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{E(M)}, \qquad x_k \in E(M),$$

for all commutative M if and only if $q_E < \infty$. Second,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E)} \approx \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(M)}, \qquad x_k \in E(M),$$

for all commutative M if and only if E is q-concave for some $q < \infty$ (see [21, Cor. 1] or [16, Thm. 1.d.6 (i) and Thm. 1.f.12 (ii)]).

We recall that if E is q-concave, then $q_E \leq q$ (see e.g. [16, p. 132]). Thus if E is q-concave for some $q < \infty$, (4.3) holds true when M is commutative. On the other hand, consider $E = L^{r,\infty}$, with $1 \leq r < \infty$. Then $p_E = q_E = r$ but E is q-concave for no finite q. Consequently the equivalence (4.3) does not hold true in general for that space E.

We do not know if (4.3) holds true for any q-concave E (with $q < \infty$) and for any M. Combining Corollary 4.2 and [19], we obtain classes of symmetric spaces having this equivalence property.

Corollary 4.3. *Suppose that either E is 2-concave, or E is 2-convex and q-concave for some $q < \infty$. Then (4.3) holds true for any M.*

The paper [9] contains two classes of spaces E satisfying (4.3). On the one hand, it is shown that all reflexive Orlicz spaces on $(0, \infty)$ have this property. On the other hand, for any $1 < p, q < \infty$ and any $\gamma \in \mathbb{R}$, the Lorentz-Zygmund space $L^{p,q}(\mathrm{Log}L)^\gamma$ is also shown to satisfy (4.3). All these spaces have non trivial Boyd indices. Combining with Theorem 1.1, we derive that for all such spaces E, we have estimates

$$\big\|(x_k)_k\big\|_{\inf} \lesssim \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E)} \lesssim \big\|(x_k)_k\big\|_{\max}, \qquad x_k \in E(M). \tag{4.4}$$

We conclude this section by showing a certain optimality of Theorem 1.1 and of (4.4). For any $p \geq 1$, we let $S^p = L^p(B(\ell^2))$ denote the Schatten p-class on ℓ^2 and we let $S^p_m = L^p(M_m)$ be its finite dimensional version. It is well-known (and easy to check) that for any $p \neq 2$,

$$\big\|(x_k)_k\big\|_{\inf} \not\approx \big\|(x_k)_k\big\|_{\max} \qquad \text{on } S^p. \tag{4.5}$$

Proposition 4.4. *Let $E = L^p(0, \infty) \cap L^q(0, \infty)$, with $1 < p < 2 < q < \infty$.*

(1) *E satisfies the equivalence property (4.3).*
(2) *There exist a semifinite von Neumann algebra M and infinite dimensional subspaces $Y, Z \subset E(M)$ such that*

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E)} \approx \big\|(x_k)_k\big\|_{\max} \qquad \textit{and} \qquad \big\|(x_k)_k\big\|_{\max} \not\approx \big\|(x_k)_k\big\|_{\inf}$$

for finite families $(x_k)_k$ of Y, whereas

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E)} \approx \big\|(x_k)_k\big\|_{\inf} \qquad \textit{and} \qquad \big\|(x_k)_k\big\|_{\inf} \not\approx \big\|(x_k)_k\big\|_{\max}$$

for finite families $(x_k)_k$ of Z.

Proof. Note that $E(M) = L^p(M) \cap L^q(M)$. Then using the Khintchine-Kahane inequality, we have for a finite family $(x_k)_k$ of $E(M)$

$$\begin{aligned}\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{E(\Omega;E(M))} &\approx \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{L^p(\Omega;L^p(M))} + \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{L^q(\Omega;L^q(M))}\\ &\approx \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(L^p(M))} + \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(L^q(M))}\\ &\approx \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(L^p(M)\cap L^q(M))}.\end{aligned}$$

This proves (1).

To show (2), let R be the hyperfinite II_1 factor. and let

$$M = R \overset{\infty}{\oplus} B(\ell^2).$$

Since the trace on R is normalized, we have $L^q(R) \subset L^p(R)$ with $\|\ \|_p \leq \|\ \|_q$ on $L^q(R)$. On the other hand, $S^p \subset S^q$ with $\|\ \|_q \leq \|\ \|_p$ on S^p. Consequently, we have a topological direct sum decomposition

$$E(M) = L^q(R) \oplus S^p.$$

Then take $Y = L^q(R) \oplus (0)$ and $Z = (0) \oplus S^p$. For any integer $m \geq 1$, there is a completely isometric embedding

$$J_m \colon S^q_m \longrightarrow L^q(R),$$

in the sense of [27] (see also [28]). In particular for any $n \geq 1$ and any $x_1, \ldots, x_n \in S^q_m$, we have

$$\big\|(J_m(x_k))_k\big\|_c = \Big\|(I_{S^q_n} \otimes J_m)\Big(\sum_{k=1}^n E_{k1} \otimes x_k\Big)\Big\|_{L^q(M_n(R))} = \Big\|\sum_{k=1}^n E_{k1} \otimes x_k\Big\|_{L^q(M_n \otimes M_m)} = \big\|(x_k)_k\big\|_c.$$

The same holds true with $\|\ \|_r$ instead of $\|\ \|_c$ and we deduce that

$$\big\|(J_m(x_k))_k\big\|_{\inf} = \big\|(x_k)_k\big\|_{\inf} \qquad \text{and} \qquad \big\|(J_m(x_k))_k\big\|_{\max} = \big\|(x_k)_k\big\|_{\max}.$$

By means of these equalities and (4.5), we obtain that Y and Z have the properties stated in the proposition. $\square$

5. Double sums

Let E and M as in Section 2. Let $(x_{ij})_{1\leq i,j\leq n}$ be a doubly indexed family of some $E(M)$. In this section we will be interested in the double Rademacher average $\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}$. Extending the definitions of Section 2, we let

$$\Big\|\sum_{i,j=1}^n \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_E$$

denote the norm of this sum in the noncommutative space $E(L^\infty(\Omega)\overline{\otimes}L^\infty(\Omega)\overline{\otimes}M)$. The analysis of this norm requires more definitions. We will use the matrix notation $[x_{ij}]$ to denote the element $\sum_{i,j=1}^n E_{ij}\otimes x_{ij}$ of $E(M_n(M))$. Accordingly we will write

$$\big\|[x_{ij}]\big\|_E = \Big\|\sum_{i,j=1}^n E_{ij}\otimes x_{ij}\Big\|_{E(M_n(M))}$$

for the norm of this matrix in $E(M_n(M))$. Also we extend the notation (1.1) to doubly indexed families by writing

$$\big\|(x_{ij})_{i,j}\big\|_c = \Big\|\Big(\sum_{i,j=1}^n x_{ij}^*x_{ij}\Big)^{\frac{1}{2}}\Big\|_{E(M)} \qquad \text{and} \qquad \big\|(x_{ij})_{i,j}\big\|_r = \Big\|\Big(\sum_{i,j=1}^n x_{ij}x_{ij}^*\Big)^{\frac{1}{2}}\Big\|_{E(M)}.$$

Finally we introduce max and inf norms as follows. First we let

$$\big\|(x_{ij})_{i,j}\big\|_{\max} = \max\big\{\big\|[x_{ij}]\big\|_E,\, \big\|[x_{ji}]\big\|_E,\, \big\|(x_{ij})_{i,j}\big\|_c,\, \big\|(x_{ij})_{i,j}\big\|_r\big\}.$$

Second we let

$$\big\|(x_{ij})_{i,j}\big\|_{\inf} = \inf\big\{\big\|[a_{ij}]\big\|_E + \big\|[b_{ji}]\big\|_E + \big\|(c_{ij})_{i,j}\big\|_c + \big\|(d_{ij})_{i,j}\big\|_r\big\},$$

where the infimum runs over all 4-tuples $(a_{ij})_{i,j}$, $(b_{ij})_{i,j}$, $(c_{ij})_{i,j}$, and $(d_{ij})_{i,j}$ of families of $E(M)$ such that $x_{ij} = a_{ij}+b_{ij}+c_{ij}+d_{ij}$ for any $1\leq i,j\leq n$.

Such norms were introduced in [12] and [27] on noncommutative L^p-spaces and the following equivalence properties hold. If $2\leq p<\infty$,

$$\Big\|\sum_{i,j}\varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_{L^p(\Omega\times\Omega;L^p(M))} \approx \big\|(x_{ij})_{i,j}\big\|_{\max}, \qquad x_{ij}\in L^p(M), \tag{5.1}$$

and if $1\leq p\leq 2$,

$$\Big\|\sum_{i,j}\varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_{L^p(\Omega\times\Omega;L^p(M))} \approx \big\|(x_{ij})_{i,j}\big\|_{\inf}, \qquad x_{ij}\in L^p(M). \tag{5.2}$$

See [12, Sect. 3] and [27, Rem. 9.8.9] for proofs and more remarks. The main purpose of this section is to extend (5.1) (resp. (5.2)) to the case when L^p is replaced by a 2-convex space such that $q_E<\infty$ (resp. a 2-concave space E). We will repeatedly use the following simple lemma.

Lemma 5.1. *For any family $(x_{ij})_{1\leq i,j\leq n}$ in $E(M)$, we have*

$$\Big\|\sum_{i,j=1}^n E_{ij}\otimes x_{ij}\Big\|_{E(M_n\otimes M)} = \Big\|\sum_{i,j=1}^n E_{i1}\otimes E_{1j}\otimes x_{ij}\Big\|_{E(M_n\otimes M_n\otimes M)}.$$

Proof. Indeed, the two elements $z_1=\sum_{i,j=1}^n E_{ij}\otimes x_{ij}$ and $z_2=\sum_{i,j=1}^n E_{i1}\otimes E_{1j}\otimes x_{ij}$ have the same distribution function, hence $\mu(z_1)=\mu(z_2)$. □

We start with a general result and the 2-convex case.

Proposition 5.2. *Assume that E is separable, or that E is the dual of a separable symmetric function space. Assume further that $q_E < \infty$ and $p_E > 1$. Then we have*

$$\big\|(x_{ij})_{i,j}\big\|_{\rm inf} \lesssim \Big\|\sum_{i,j} \varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E \lesssim \big\|(x_{ij})_{i,j}\big\|_{\rm max}$$

for finite families $(x_{ij})_{i,j}$ of $E(M)$.

Proof. The upper estimate is a simple reiteration argument. For any $x_{ij}\in E(M)$, we write

$$\sum_{i,j} \varepsilon_i\otimes\varepsilon_j\otimes x_{ij} = \sum_i \varepsilon_i\otimes z_i,\qquad\text{with}\qquad z_i=\sum_j \varepsilon_j\otimes x_{ij},$$

and we apply the upper estimate of Theorem 1.1 on $L^\infty(\Omega)\overline{\otimes}M$. We find that

$$\begin{aligned}\Big\|\sum_{i,j} \varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E &\lesssim \Big\|\sum_i E_{1i}\otimes z_i\Big\|_E + \Big\|\sum_i E_{i1}\otimes z_i\Big\|_E\\ &= \Big\|\sum_j \varepsilon_j\otimes\Big(\sum_i E_{1i}\otimes x_{ij}\Big)\Big\|_E + \Big\|\sum_j \varepsilon_j\otimes\Big(\sum_i E_{i1}\otimes x_{ij}\Big)\Big\|_E.\end{aligned}$$

Applying Theorem 1.1 (2) again, together with Lemma 5.1, we see that

$$\begin{aligned}\Big\|\sum_j \varepsilon_j\otimes\Big(\sum_i E_{1i}\otimes x_{ij}\Big)\Big\|_E &\lesssim \Big\|\sum_{ij} E_{1j}\otimes E_{1i}\otimes x_{ij}\Big\|_E + \Big\|\sum_{ij} E_{j1}\otimes E_{1i}\otimes x_{ij}\Big\|_E\\ &= \big\|(x_{ij})_{i,j}\big\|_r + \big\|[x_{ji}]\big\|_E.\end{aligned}$$

Likewise,

$$\Big\|\sum_j \varepsilon_j\otimes\Big(\sum_i E_{i1}\otimes x_{ij}\Big)\Big\|_E \lesssim \big\|(x_{ij})_{i,j}\big\|_c + \big\|[x_{ij}]\big\|_E.$$

Combined with the previous inequality, these yield the desired upper estimate.

The lower estimate can be deduced from the upper one by duality, the argument being similar to the one in the proof of Theorem 1.1 given in Section 3. We skip the details. □

Theorem 5.3. *Assume that E is separable, or that E is the dual of a separable symmetric function space. If $E\in{\rm Int}(L^2,L^q)$ for some $q<\infty$, then we have*

$$\Big\|\sum_{i,j} \varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E \approx \big\|(x_{ij})_{i,j}\big\|_{\rm max}$$

for finite families $(x_{ij})_{ij}$ of $E(M)$.

In particular, this holds true if $q_E<\infty$ and if either E is 2-convex or $p_E>2$.

Proof. Assume that $E\in{\rm Int}(L^2,L^q)$ for some $q<\infty$. The estimate $\lesssim$ is given by Proposition 5.2. As in the proof of Corollary 4.1, the reverse estimate is proved by interpolation, using (5.1) on L^q.

The last line of the statement then follows from [13, Thm. 7.3]. □

We shall now consider double sums in the 2-concave case or, more generally, in the case when $E\in{\rm Int}(L^1,L^2)$. This case turns out to be much more delicate than the 2-convex one.

The major obstacle is that we do not know whether the lower estimate in Proposition 5.2 remains true in the case when $1 \leq p_E \leq q_E < \infty$.

We will need to somehow replace the Rademacher functions by the generators of a free group living in the associated group von Neumann algebra. The use of such techniques goes back to Haagerup and Pisier [12]. In the sequel, we let $G = \mathbb{F}_\infty$ be a free group with an infinite sequence $\gamma_1, \ldots, \gamma_n, \ldots$ of generators. Let $(\delta_g)_{g\in G}$ denote the canonical basis of ℓ^2_G and let $\lambda\colon G \to B(\ell^2_G)$ be the left regular representation of G, defined by

$$\lambda(g)\delta_h = \delta_{gh}, \qquad g, h \in G.$$

We recall that the group von Neumann algebra of G is defined as

$$VN(G) = \big\{\lambda(g) \,:\, g \in G\big\}'' = \overline{\mathrm{Span}}^{w^*}\big\{\lambda(g) \,:\, g \in G\big\} \subset B(\ell^2_G).$$

For simplicity we let $\mathcal{M} = VN(G)$ in the sequel. Let e be the unit element of G. Then $\mathcal{M}$ has a canonical normalized normal trace σ defined by $\sigma(z) = \langle z(\delta_e), \delta_e\rangle$.

Let (M, τ) be an arbitrary semifinite von Neumann algebra. In the sequel we regard the von Neumann tensor product $\mathcal{M}\overline{\otimes}M$ as equipped with $\sigma \otimes \tau$ in the usual way. It is remarkable that the noncommutative Khintchine inequalities on L^p remain unchanged if one replaces the Rademacher sequence by the $\lambda(\gamma_k)$'s. Namely for any $1 \leq p < \infty$, there is an equivalence

$$\Big\|\sum_k \lambda(\gamma_k)\otimes x_k\Big\|_{L^p(\mathcal{M}\overline{\otimes}M)} \approx \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{L^p(\mathbb{T};L^p(M))} \tag{5.3}$$

for finite families $(x_k)_k$ of $L^p(M)$ (see [12, Sect. 3] and [27, Thm. 9.8.7]).

Let $x_1, \ldots, x_n \in E(M)$. It is clear that each $\lambda(\gamma_k)\otimes x_k$ belongs to $E(\mathcal{M}\overline{\otimes}M)$ and we write

$$\Big\|\sum_k \lambda(\gamma_k)\otimes x_k\Big\|_E$$

for the norm of their sum $\sum_k \lambda(\gamma_k)\otimes x_k$ in the latter space.

Let $a \in \mathcal{M}$ and let $\varphi_a\colon L^1(\mathcal{M}) \to \mathbb{C}$ be the functional defined by $\varphi_a(z) = \sigma(za)$ for any $z \in L^1(\mathcal{M})$. The algebraic tensor product $L^1(\mathcal{M}) \otimes L^1(M)$ is dense in $L^1(\mathcal{M}\overline{\otimes}M)$ (see e.g. [10, Sect. 3]) and $\varphi_a \otimes I_{L^1(M)}$ uniquely extends to a bounded operator $T_a^1\colon L^1(\mathcal{M}\overline{\otimes}M) \to L^1(M)$. Indeed, this extension is the pre-adjoint of the embedding $M \to \mathcal{M}\overline{\otimes}M$ taking any $x \in M$ to $a \otimes x$. Likewise, $\varphi_{a|\mathcal{M}} \otimes I_M$ uniquely extends to a contractive normal operator $T_a^\infty\colon \mathcal{M}\overline{\otimes}M \to M$. Moreover these two maps coincide on the intersection of $L^1(\mathcal{M}\overline{\otimes}M)$ and $\mathcal{M}\overline{\otimes}M$. Hence we may define a bounded linear map

$$T_a\colon L^1(\mathcal{M}\overline{\otimes}M) + \mathcal{M}\overline{\otimes}M \longrightarrow L^1(M) + M$$

extending both of them. In the sequel it will be convenient to write

$$\langle u, a\rangle = T_a(u), \qquad u \in L^1(\mathcal{M}\overline{\otimes}M) + \mathcal{M}\overline{\otimes}M.$$

Fix an integer $n \geq 1$ and let

$$P_n\colon L^1(\mathcal{M}\overline{\otimes}M) + \mathcal{M}\overline{\otimes}M \longrightarrow L^1(\mathcal{M}\overline{\otimes}M) + \mathcal{M}\overline{\otimes}M$$

be defined by

$$P_n(u) = \sum_{k=1}^{n} \lambda(\gamma_k) \otimes \langle u, \lambda(\gamma_k^{-1})\rangle\,. \tag{5.4}$$

This is a projection which extends the orthogonal projection $L^2(\mathcal{M}\overline{\otimes}M) \to L^2(\mathcal{M}\overline{\otimes}M)$ onto the subspace $\mathrm{Span}\{\lambda(\gamma_k)\,:\, 1 \leq k \leq n\} \otimes L^2(M)$.

Lemma 5.4. *There exist a constant $K_E \geq 0$ such that for any $n \geq 1$ and any (M, τ) as above,*

$$\big\| P_n \colon E(\mathcal{M}\overline{\otimes}M) \longrightarrow E(\mathcal{M}\overline{\otimes}M)\big\| \leq K_E.$$

Proof. By construction, P_n is the extension of two bounded maps

$$P_n^\infty \colon \mathcal{M}\overline{\otimes}M \longrightarrow \mathcal{M}\overline{\otimes}M \qquad \text{and} \qquad P_n^1 \colon L^1(\mathcal{M}\overline{\otimes}M) \longrightarrow L^1(\mathcal{M}\overline{\otimes}M).$$

Since the L^2-realization of P_n is selfadjoint, P_n^∞ is the adjoint of the mapping $v \mapsto [P_n^1(v^*)]^*$ on $L^1(\mathcal{M}\overline{\otimes}M)$. Hence

$$\|P_n^\infty\| = \|P_n^1\|. \tag{5.5}$$

Let $\mathcal{P} = \mathrm{Span}\{\lambda(g)\,:\, g \in G\}$. According to [12, Prop. 1.1], the restriction of P_n^∞ to $\mathcal{P} \otimes M$ has norm ≤ 2. Further, $\mathcal{P} \otimes M$ is a w^*-dense $*$-subalgebra of $\mathcal{M}\overline{\otimes}M$. Hence the unit ball of $\mathcal{P} \otimes M$ is w^*-dense in the unit ball of $\mathcal{M}\overline{\otimes}M$ by Kaplansky's Theorem. Since P_n^∞ is w^*-continuous, we deduce that $\|P_n^\infty\| \leq 2$. The result now follows from (5.5) and Proposition 2.1. $\square$

Lemma 5.5. *Assume that $q_E < \infty$. Then we have an estimate*

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \lesssim \Big\|\sum_k \lambda(\gamma_k) \otimes x_k\Big\|_E$$

for finite families $(x_k)_k$ of $E(M)$.

Proof. For any $n \geq 1$, let us define

$$Q_n \colon L^1(\mathcal{M}\overline{\otimes}M) + \mathcal{M}\overline{\otimes}M \longrightarrow L^1(\mathbb{T}; L^1(M)) + L^\infty(\mathbb{T})\overline{\otimes}M$$

in a similar way to P_n, by letting

$$Q_n(u) = \sum_{k=1}^{n} \varepsilon_k \otimes \langle u, \lambda(\gamma_k^{-1})\rangle\,.$$

Applying Lemma 5.4 with $E = L^p$ and (5.3), we obtain that for any $1 \leq p < \infty$, we have

$$\big\| Q_n \colon L^p(\mathcal{M}\overline{\otimes}M) \longrightarrow L^p(\mathbb{T}; L^p(M))\big\| \leq K_p$$

for some constant K_p only depending on p. Since $q_E < \infty$, there exists some $1 < q < \infty$ such that $E \in \mathrm{Int}(L^1, L^q)$. Applying the above estimate with $p = 1$ and $p = q$ together with Proposition 2.1, we deduce that

$$\big\| Q_n \colon E(\mathcal{M}\overline{\otimes}M) \longrightarrow E(L^\infty(\mathbb{T})\overline{\otimes}M)\big\| \leq D_E$$

for some constant D_E only depending on E. Since

$$Q_n\Big(\sum_{k=1}^{n} \lambda(\gamma_k)\otimes x_k\Big) \,=\, \sum_{k=1}^{n} \varepsilon_k \otimes x_k$$

for any $x_1, \ldots, x_n$ in $E(M)$, we obtain the desired estimate. □

Using K-convexity as in Remark 2.4, it is not hard to see that if $p_E > 1$ and $q_E < \infty$, then the two averages $\big\|\sum_k \lambda(\gamma_k)\otimes x_k\big\|_E$ and $\big\|\sum_k \varepsilon_k\otimes x_k\big\|_E$ are actually equivalent. We do not know if this equivalence holds true if we merely assume that $q_E < \infty$. However we have the following special case.

Lemma 5.6. *Assume that $E \in \mathrm{Int}(L^1, L^2)$. Then we have an equivalence*

$$\Big\|\sum_k \lambda(\gamma_k)\otimes x_k\Big\|_E \,\approx\, \Big\|\sum_k \varepsilon_k\otimes x_k\Big\|_E \Big(\approx\, \|(x_k)_k\|_{\mathrm{inf}}\Big)$$

for finite families $(x_k)_k$ of $E(M)$.

Proof. Indeed arguing as in the proof of Corollary 4.1 (1) and using equivalence (5.3), we see that

$$\Big\|\sum_k \lambda(\gamma_k)\otimes x_k\Big\|_E \,\lesssim\, \big\|(x_k)_k\big\|_{\mathrm{inf}}.$$

Combining with Lemma 5.5 and the lower estimate in Theorem 1.1, one gets the equivalence. □

Theorem 5.7. *Assume that E is separable, or that E is the dual of a separable symmetric function space. If $E \in \mathrm{Int}(L^1, L^2)$, then we have*

$$\Big\|\sum_{i,j} \varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E \,\approx\, \big\|(x_{ij})_{i,j}\big\|_{\mathrm{inf}}$$

for finite families $(x_{ij})_{ij}$ of $E(M)$.

In particular, this holds true if either E is 2-concave or $q_E < 2$.

Proof. By [13, Thm. 7.3], the last assertion will follow from the main one. Thus we assume that $E \in \mathrm{Int}(L^1, L^2)$. Then the estimate

$$\Big\|\sum_{i,j=1}^{n} \varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E \,\lesssim\, \big\|(x_{ij})_{i,j}\big\|_{\mathrm{inf}}$$

follows from interpolation principles as in Corollary 4.1 (1), using (5.2) for $p = 1$. We are now going to concentrate on the converse inequality.

We first observe that

$$\Big\|\sum_{i,j} \lambda(\gamma_i)\otimes\lambda(\gamma_j)\otimes x_{ij}\Big\|_E \,\approx\, \Big\|\sum_{i,j} \varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E \tag{5.6}$$

for finite doubly indexed families $(x_{ij})_{i,j}$ of $E(M)$, where $\|\cdots\|_E$ in the left hand-side stands for the norm of the double sum $\sum_{i,j}\lambda(\gamma_i)\otimes\lambda(\gamma_j)\otimes x_{ij}$ in the space $E(\mathcal{M}\overline{\otimes}\mathcal{M}\overline{\otimes}M)$. Indeed

applying Lemma 5.6 first on $L^\infty(\Omega)\overline{\otimes}M$ and then on $\mathcal{M}\overline{\otimes}M$, we have

$$\begin{aligned}\Big\|\sum_i \varepsilon_i\otimes\Big(\sum_j \varepsilon_j\otimes x_{ij}\Big)\Big\|_E &\approx \Big\|\sum_i \lambda(\gamma_i)\otimes\Big(\sum_j \varepsilon_j\otimes x_{ij}\Big)\Big\|_E\\ &= \Big\|\sum_j \varepsilon_j\otimes\Big(\sum_i \lambda(\gamma_i)\otimes x_{ij}\Big)\Big\|_E\\ &\approx \Big\|\sum_j \lambda(\gamma_j)\otimes\Big(\sum_i \lambda(\gamma_i)\otimes x_{ij}\Big)\Big\|_E,\end{aligned}$$

which yields (5.6). It therefore suffices to show an estimate

$$\Big\|\sum_{i,j}\lambda(\gamma_i)\otimes\lambda(\gamma_j)\otimes x_{ij}\Big\|_E \gtrsim \big\|(x_{ij})_{i,j}\big\|_{\rm inf}. \tag{5.7}$$

Fix an integer $n\geq 1$ and let $(x_{ij})_{1\leq i,j\leq n}$ be a family of $E(M)$. We write

$$\sum_{i,j=1}^n \lambda(\gamma_i)\otimes\lambda(\gamma_j)\otimes x_{ij} = \sum_{i=1}^n \lambda(\gamma_i)\otimes z_i,\qquad\text{with}\qquad z_i=\sum_{j=1}^n\lambda(\gamma_j)\otimes x_{ij}.$$

Then according to Lemma 5.6, there exist $z_1',\ldots,z_n'$, $z_1'',\ldots,z_n''$ in $E(\mathcal{M}\overline{\otimes}M)$ such that $z_i=z_i'+z_i''$ for any i and

$$\Big\|\sum_{i=1}^n E_{i1}\otimes z_i'\Big\|_{E(M_n\otimes\mathcal{M}\overline{\otimes}M)} \lesssim \Big\|\sum_{i,j}\lambda(\gamma_i)\otimes\lambda(\gamma_j)\otimes x_{ij}\Big\|_E,$$

$$\Big\|\sum_{i=1}^n E_{1i}\otimes z_i''\Big\|_{E(M_n\otimes\mathcal{M}\overline{\otimes}M)} \lesssim \Big\|\sum_{i,j}\lambda(\gamma_i)\otimes\lambda(\gamma_j)\otimes x_{ij}\Big\|_E.$$

Let P_n be defined by (5.4). We clearly have $P_n(z_i)=z_i$, hence $z_i=P_n(z_i')+P_n(z_i'')$ for any $i=1,\ldots,n$. Moreover there exist two families $(u_{ij})_{1\leq i,j\leq n}$ and $(v_{ij})_{1\leq i,j\leq n}$ of $E(M)$ such that

$$P_n(z_i')=\sum_{j=1}^n\lambda(\gamma_j)\otimes u_{ij}\qquad\text{and}\qquad P_n(z_i'')=\sum_{j=1}^n\lambda(\gamma_j)\otimes v_{ij}$$

for any i. Then we have decompositions

$$x_{ij}=u_{ij}+v_{ij},\qquad 1\leq i,j\leq n.$$

Let us now apply Lemma 5.4 on $M_n\otimes M$, with $I_{M_n}\otimes P_n$ instead of P_n. We find that

$$\begin{aligned}\Big\|\sum_{j=1}^n\lambda(\gamma_j)\otimes\Big(\sum_{i=1}^n E_{i1}\otimes u_{ij}\Big)\Big\|_E &= \Big\|\sum_{i=1}^n E_{i1}\otimes P_n(z_i')\Big\|_E\\ &\lesssim \Big\|\sum_{i=1}^n E_{i1}\otimes z_i'\Big\|_E\\ &\lesssim \Big\|\sum_{i,j}\lambda(\gamma_i)\otimes\lambda(\gamma_j)\otimes x_{ij}\Big\|_E.\end{aligned}$$

In the same manner,

$$\Big\|\sum_{j=1}^{n}\lambda(\gamma_j)\otimes\Big(\sum_{i=1}^{n}E_{1i}\otimes v_{ij}\Big)\Big\|_E \lesssim \Big\|\sum_{i,j}\lambda(\gamma_i)\otimes\lambda(\gamma_j)\otimes x_{ij}\Big\|_E.$$

We can now repeat the above arguments, using the projection Col_n and (2.3) (as well as its row counterpart) instead of P_n. We therefore obtain new families $(a_{ij})_{i,j}$, $(b_{ij})_{i,j}$, $(c_{ij})_{i,j}$, and $(d_{ij})_{i,j}$ in $E(M)$ such that

$$u_{ij}=c_{ij}+a_{ij},\qquad v_{ij}=b_{ij}+d_{ij},\qquad 1\leq i,j\leq n,$$

and

$$\Big\|\sum_{i,j}E_{j1}\otimes E_{i1}\otimes c_{ij}\Big\|_E \lesssim \Big\|\sum_{j}\lambda(\gamma_j)\otimes\Big(\sum_{i}E_{i1}\otimes u_{ij}\Big)\Big\|_E,$$

$$\Big\|\sum_{i,j}E_{1j}\otimes E_{i1}\otimes a_{ij}\Big\|_E \lesssim \Big\|\sum_{j}\lambda(\gamma_j)\otimes\Big(\sum_{i}E_{i1}\otimes u_{ij}\Big)\Big\|_E,$$

$$\Big\|\sum_{i,j}E_{j1}\otimes E_{1i}\otimes b_{ij}\Big\|_E \lesssim \Big\|\sum_{j}\lambda(\gamma_j)\otimes\Big(\sum_{i}E_{1i}\otimes v_{ij}\Big)\Big\|_E,$$

$$\Big\|\sum_{i,j}E_{1j}\otimes E_{1i}\otimes d_{ij}\Big\|_E \lesssim \Big\|\sum_{j}\lambda(\gamma_j)\otimes\Big(\sum_{i}E_{1i}\otimes v_{ij}\Big)\Big\|_E.$$

According to Lemma 5.1, these estimate imply that the four quantities

$$\big\|[a_{ij}]\big\|_E,\ \ \big\|[b_{ji}]\big\|_E,\ \ \Big\|\Big(\sum_{ij}c_{ij}^*c_{ij}\Big)^{\frac{1}{2}}\Big\|_E,\ \ \text{and}\ \Big\|\Big(\sum_{ij}d_{ij}d_{ij}^*\Big)^{\frac{1}{2}}\Big\|_E$$

are all

$$\lesssim\Big\|\sum_{i,j}\lambda(\gamma_i)\otimes\lambda(\lambda_j)\otimes x_{ij}\Big\|_E.$$

Furthermore we have $x_{ij}=u_{ij}+v_{ij}=a_{ij}+b_{ij}+c_{ij}+d_{ij}$ for any i,j, hence we obtain that (5.7) holds true. □

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